1.(i) (a) The line M through the point $Q(4, 2, 0)$ and in the direction $\vec{v} = 3\vec{i} - \vec{j} + \vec{k}$ has the parametric form

$$x = 4 + 3t; \quad y = 2 - t; \quad z = t. \qquad (A2)$$

Note: $\vec{r} = (4, 2, 0) + t(3, -1, 1)$ is also acceptable.

(b) Since any point R on the line M can be written as $(4 + 3t, 2 - t, t)$ for some value of t then the vector $\vec{w} = \vec{RP}$ is

$$\begin{aligned} \vec{w} &= (2 - 4 - 3t)\vec{i} + (3 - 2 + t)\vec{j} + (1 - t)\vec{k} \\ &= (-2 - 3t)\vec{i} + (1 + t)\vec{j} + (1 - t)\vec{k} \end{aligned} \qquad (M1)\ (A1)$$

(c) The cross product $\vec{u} \times \vec{w}$ can be written as

$$\vec{u} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ (-2 - 3t) & (1 + t) & (1 - t) \end{vmatrix}$$

$$\begin{aligned} &= \{2(1 - t) + 3(1 + t)\vec{i} - \{(1 - t) + 3(-2 - 3t)\}\vec{j} \\ &\qquad + \{(1 + t) - 2(-2 - 3t)\}\vec{k} \\ &= (5 + t)\vec{i} + 5(1 + 2t)\vec{j} + (5 + 7t)\vec{k} \end{aligned}$$

Thus

$$\begin{aligned} |\vec{u} \times \vec{w}| &= \sqrt{(5 + t)^2 + 25(1 + 2t)^2 + (5 + 7t)^2} \\ &= \sqrt{75 + 180t + 150t^2} = \sqrt{15(5 + 12t + 10t^2)} \end{aligned} \qquad (M2)\ (A2)$$

(d) The value of $|\vec{u} \times \vec{w}|$ is minimised when $\dfrac{d}{dt}(|\vec{u} \times \vec{w}|) = 0$.
Thus (Note 1)

$$\frac{d}{dt}\left\{\sqrt{15(5 + 12t + 10t^2}\right\} = \frac{1}{2} \times \left\{\frac{15(12 + 20t)}{\sqrt{15(5 + 12t + 10t^2)}}\right\} = 0$$

and so this will be so when $12 + 20t = 0$, or $t = -\dfrac{3}{5}$.

The minimum value is then $\sqrt{15\left[5 + 12\left(-\dfrac{3}{5}\right) + 10\left(-\dfrac{3}{5}\right)^2\right]} = \sqrt{21}$. $\qquad$ (M2) (A2)

(ii) In polar form,

$$-i = e^{3i\pi/2} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$$

and so the cube roots of $-i$ are

$$z_k = e^{(2k\pi + 3\pi/2)i/3}, \quad k = 0, 1, 2,$$

and so

$$z_0 = e^{i\pi/2} = i; \; z_1 = e^{7i\pi/6} = -\frac{\sqrt{3}}{2} - \frac{i}{2};$$ *(A3)*

$$z_2 = e^{11i\pi/6} = \frac{\sqrt{3}}{2} - \frac{i}{2}$$ (Note 2)

Then

$$[(1 - i)z]^3 + i = 0 \Rightarrow z = \frac{1}{1 - i}\sqrt[3]{-i} = \frac{1}{1 - i}z_k, \quad k = 0, 1, 2.$$

Hence the roots are

$$\frac{i}{1 - i}, \quad \frac{1}{1 - i}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right), \quad \frac{1}{1 - i}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

or

$$\frac{i - 1}{2}, \quad \frac{1 - \sqrt{3} - i(\sqrt{3} + 1)}{4}, \quad \frac{1 + \sqrt{3} + i(\sqrt{3} - 1)}{4}$$ *(M2) (A3)*

Note 1: Other methods are acceptable. One such solution is to consider that $|\vec{u} \times \vec{w}|$ is minimised when $10t^2 + 12t + 5$ is minimised. This in turn is a parabola. The minimum is at $x = -\frac{b}{2a} = -\frac{3}{5}$ and hence the result.

Note 2: The algebraic forms of $\sqrt[3]{-i}$ given above, on the same line as the exponential forms, are not needed at that stage and the *A3* marks on line 6 do not depend on them being given there. They are of course required later.

2. (a) If the function $x(t)$ is defined by

$$x(t) = e^{-\lambda t}\sin(pt + \alpha)$$

then
$$\frac{dx}{dt} = -\lambda e^{-\lambda t}\sin(pt + \alpha) + pe^{-\lambda t}\cos(pt + \alpha)$$
$$= -\lambda x + pe^{-\lambda t}\cos(pt + \alpha) \text{ (Note 1)}$$ *(M1) (A1)*

Differentiating again

$$\frac{d^2x}{dt^2} = -\lambda\frac{dx}{dt} + p\{-\lambda e^{-\lambda t}\cos(pt + \alpha) - pe^{-\lambda t}\sin(pt + \alpha)\}$$
$$= -\lambda\frac{dx}{dt} - \lambda pe^{-\lambda t}\cos(pt + \alpha) - p^2x$$
$$= -\lambda\frac{dx}{dt} - \lambda\left\{\frac{dx}{dt} + \lambda x\right\} - p^2x$$
$$= -2\lambda\frac{dx}{dt} - (\lambda^2 + p^2)x$$ *(M1) (A1)*

Thus

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + (\lambda^2 + p^2)x = 0$$ *(R2)(AG)*

(b) When $\alpha = 0$, $x = e^{-\lambda t}\sin pt$ and so $\frac{dx}{dt} = -\lambda e^{-\lambda t}\sin pt + pe^{-\lambda t}\cos pt$.

Then, when $t = \frac{\pi}{p}$, $\sin pt = 0$ and $\cos pt = -1$. Hence

$$-pe^{-\lambda\pi/p} = -2p \Rightarrow e^{-\lambda\pi/p} = 2$$ *(R1)*

Also

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + (\lambda^2 + p^2)x = -3p - 4\lambda p = 0$$ *(R1)*

since $x = 0$.

Therefore $\lambda = -\frac{3}{4}$ and then $e^{3\pi/4p} = 2 \Rightarrow \frac{3\pi}{4p} = \log_e 2$

and so $p = \frac{3\pi}{4\log_e 2}$ *(R2)(AG)*

(c) (i) As above, $x(t) = e^{-\lambda t}\sin(pt + \alpha)$,

$$\Rightarrow \frac{dx}{dt} = -\lambda e^{-\lambda t}\sin(pt + \alpha) + pe^{-\lambda t}\cos(pt + \alpha)$$

and as $\frac{dx}{dt} = 0$ when $t = 0$ it follows that

$$\lambda\sin\alpha = p\cos\alpha \Rightarrow \lambda = p\cot\alpha$$ *(R2)(AG)*

(ii) The other values of t for which $\frac{dx}{dt}$ is zero are given by

$$\lambda \sin(pt + \alpha) = p\cos(pt + \alpha)$$

or $$\tan(pt + \alpha) = \frac{p}{\lambda} = \tan\alpha$$

and this is satisfied when $t = 0 + \frac{k\pi}{p}$, $k = 1, 2, 3, 4, \ldots$

These values are in arithmetic progression with common difference $\frac{\pi}{p}$. *(R2) (A2)*

(iii) From (ii), two consecutive values of t when $\frac{dx}{dt} = 0$ are $\frac{k\pi}{p}$ and $\frac{(k+1)\pi}{p}$. *(A1)*

The corresponding values of x are

$$x_k = e^{-\lambda k\pi/p}\sin(k\pi + \alpha) = (-1)^k e^{-\lambda k\pi/p}\sin\alpha$$

and

$$x_{k+1} = e^{-\lambda(k+1)\pi/p}\sin((k+1)\pi + \alpha)$$
$$= (-1)^{k+1}e^{-\lambda(k+1)\pi/p}\sin\alpha$$ *(R2)*

Thus

$$\frac{x_{k+1}}{x_k} = -e^{-\lambda\pi/p} = -e^{-\pi\cot\alpha}$$

and this is a constant. Hence the values of x at which $\frac{dx}{dt} = 0$ are in geometric progression with common ratio $-e^{-\pi\cot\alpha}$. *(R2) (A1)*

Note 1: Students may present their results in other forms.

3. (i) Since the curve of $x^2\sin x$ cuts the axis at 0, π and 2π, the area is given by

$$\int_0^{2\pi} |x^2\sin x|dx = \int_0^{\pi} x^2\sin x\,dx - \int_{\pi}^{2\pi} x^2\sin x\,dx \qquad (A2)$$

Integration by parts gives

$$\begin{aligned}\int x^2\sin x\,dx &= -x^2\cos x + 2\int x\cos x\,dx \\ &= -x^2\cos x + 2\left\{x\sin x - \int\sin x\,dx\right\} \\ &= -x^2\cos x + 2x\sin x + 2\cos x + C \qquad (M2)\ (A2)\end{aligned}$$

Hence the area is

$$\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi} - \left[-x^2\cos x + 2x\sin x + 2\cos x\right]_{\pi}^{2\pi}$$
$$= \{-(\pi^2)(-1) - 2 - 2\} + |-4\pi^2 + 2 - (\pi^2 - 2)|$$
$$= \pi^2 - 4 + |-5\pi^2 + 4| = \pi^2 - 4 + 5\pi^2 - 4 = 6\pi^2 - 8 \qquad (M2)\ (A2)$$

Note: Some candidates may simply evaluate the integral $\int_0^{2\pi} x^2\sin x\,dx$ and obtain the result $-4\pi^2$. If so award *M2, A2* for a correct integration by parts and *M1, A1* for obtaining $-4\pi^2$. The *A2* marks at the start would not be awarded.

(ii) (a) Rearranging the equations (Note 1)

$$\begin{aligned}-3x + y + 2z &= a \quad (1)\\ -11x + 2y + 6z &= b \quad (2)\\ 7x + y - 2z &= c \quad (3)\end{aligned}$$

as

$$\begin{aligned}y + 2z - 3x &= a\\ 2y + 6z - 11x &= b\\ y - 2z + 7x &= c\end{aligned}$$

and eliminating y from the last two equations gives

$$\begin{aligned}y + 2z - 3x &= a\\ 2z - 5x &= b - 2a\\ -4z + 10x &= c - a\end{aligned}$$

Now eliminate z from the last equation to give

$$\begin{aligned}y + 2z - 3x &= a\\ 2z - 5x &= b - 2a\\ 0 &= c + 2b - 5a\end{aligned}$$

Hence the system will only have a solution if $5a - 2b - c = 0$. (M3) (A2)

(b) If $a = 2$ and $b = 7$ then c must equal -4 for there to be a solution, otherwise there are no solutions. (Note 2)

Then, with $a = 2$ and $b = 7$,

$$2z - 5x = b - 2a = 7 - 4 = 3$$

and $$y + 2z - 3x = a = 2.$$

A solution of the first of these two equations is $z = 4$, $x = 1$ and substituting these in the second equation yields $y = -3$. *(A3)*

Since the system reduces to two equations on three unknowns, reflected by the fact that the determinant of the coefficients is zero, there will be an infinity of solutions, namely $x = \lambda$, $y = -1 - 2\lambda$, $z = \frac{1}{2}(3 + 5\lambda)$. *(R1) (A1)*

Note 1: Other approaches are possible. One such solution is to use Gaussian elimination.

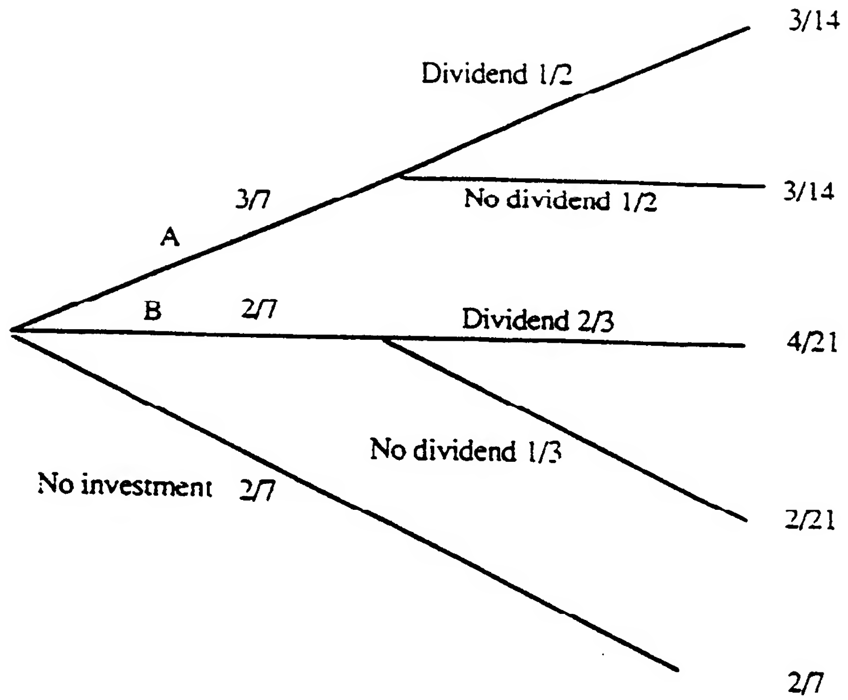
Note 2: Students may choose to solve this equation by elimination again.

OR: (1) + (3) $4x + 2y = a + c = -2$

(2) + 3(3) $10x + 5y = b + 3c = -5$

which gives the solution in terms of either x or y as parameters.

4. (a)



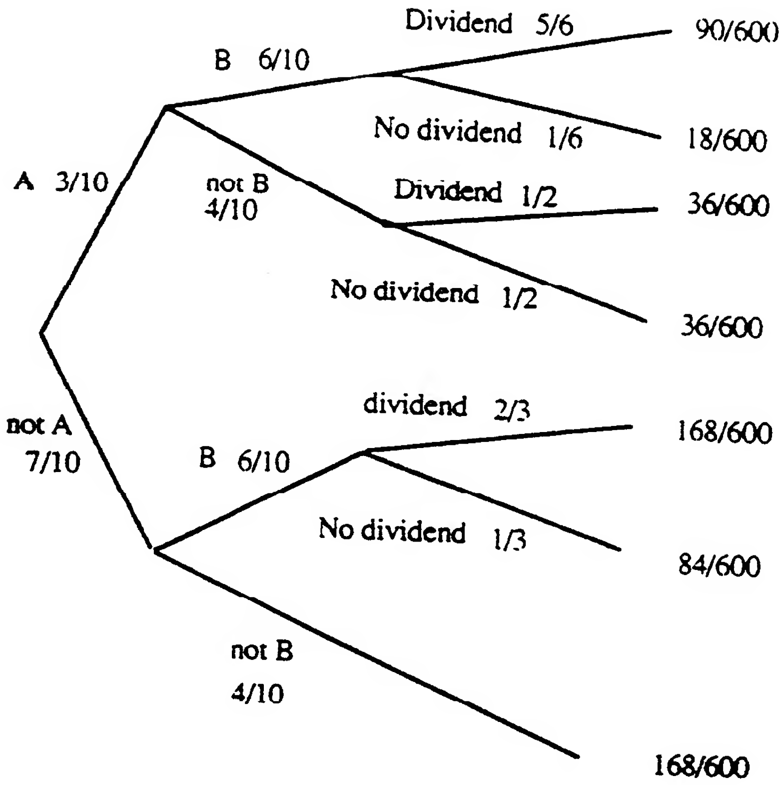
(A3)
less 1 per error
in final column.
Follow through
if error occurs.

Probability of receiving a dividend $= \dfrac{3}{14} + \dfrac{4}{21} = \dfrac{17}{42}$. *(A2)*

Probability that it came from A

$$= \frac{p(\text{dividend and invested in } A)}{p(\text{dividend})} = \frac{3/14}{17/42} = \frac{9}{17}$$ *(R2) (A1)*

(b) (i)



(A3) less 1 per error in final column. Follow through if errors occur.

(ii) If she invests in both companies,

probability of no dividend $= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

and so

probability of dividend from at least one

$= 1 - \frac{1}{6} = \frac{5}{6}$. *(A2)*

(iii) Probability of no investment $= \frac{7}{10} \times \frac{4}{10} = \frac{28}{100} = \frac{7}{25}$. *(A1)*

(iv) Probability of a dividend

$$= \frac{90}{600} + \frac{36}{600} + \frac{168}{600} = \frac{294}{600} = \frac{49}{100}$$

and so

Probability of no dividend $= 1 - \frac{294}{600} = \frac{306}{600} = \frac{51}{100}$. *(A2)*

Finally

Probability {no investment made given no dividend}

$$= \frac{\frac{168}{600}}{\frac{306}{600}} = \frac{168}{306} = \frac{28}{51}.$$ *(A2)*

5. (i) If a and b are real numbers then $a * b = \sqrt[3]{a^3 + b^3}$ is a real number and so **R** is closed under $*$. *(R2)*

Then, for any real number a, b and c

$$a * (b * c) = a * \left(\sqrt[3]{b^3 + c^3}\right) = \sqrt[3]{a^3 + \left(\sqrt[3]{(b^3 + c^3)}\right)^3}$$
$$= \sqrt[3]{a^3 + b^3 + c^3}$$

Similarly

$$(a * b) * c = \sqrt[3]{a^3 + b^3 + c^3}$$

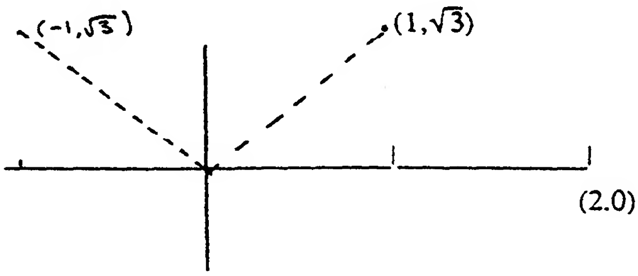
and so the set **R** is associative under $*$. *(R3)*

Let e be the identity element, if one exists. Then
$a * e = e * a = a$ for all a in **R**.

But $a * e = a \Rightarrow \sqrt[3]{a^3 + e^3} = a \Rightarrow a^3 + e^3 = a^3$ or $e = 0$.
Hence there is an identity element, zero. *(R3)*

Finally, each element a in **R** must have an inverse a^{-1} such that $a * a^{-1} = e$.
That is $\sqrt[3]{a^3 + (a^{-1})^3} = 0$ and this is satisfied by $a^{-1} = -a$. *(R3)*

(ii) (a)



The set of rotations about the origin which map any one of these points onto itself or onto one of the other points is

$S = \{R_0, R_{\pi/3}, R_{2\pi/3}, R_{4\pi/3}, R_{5\pi/3}\}$ *(A3)*, less 1 per error

(b) The set S is not a group under the composition of rotations because

$$R_{2\pi/3} * R_{\pi/3} = R_\pi$$

and R_π is not a member of the set S.

Hence S is not closed under composition and so S is not a group under this operation. *(R3)*

(c) Add the rotation R_π to the set S and the new set $S \cup R_\pi$ does form a group under composition. *(A2)*

Then, since

$$R^2_{\pi/3} = R_{2\pi/3}, \quad R^3_{\pi/3} = R_\pi, \quad R^4_{\pi/3} = R_{4\pi/3}$$

and

$$R^5_{\pi/3} = R_{5\pi/3}, \quad R^6_{\pi/3} = R_{6\pi/3} = R_0$$

the group is cyclic with generator $R_{\pi/3}$. (The element $R_{5\pi/3}$ is also a generator.) *(R4)*

There are two proper subgroups and they are

$$\{R_0,\ R_\pi\} \text{ and } \{R_0,\ R_{2\pi/3},\ R_{4\pi/3}\}.$$

(A1) (A1)

(iii) (a) Two groups $\{G_1, *\}$ and $\{G_2, \circ\}$ are **isomorphic** if there is a *bijection*, *i.e.* a function $f\colon G_1 \to G_2$ that is one-to-one and onto, such that

$$f(a_1 * a_2) = f(a_1) \circ f(a_2)$$

for all a_1 and a_2 in G_1. (There will be a variety of ways of writing the above). *(A4)*

(b)(1) Let $a_2 = f(a_1)$, and e_1 the identity in G_1 and e_2 the identity in G_2.

$\Rightarrow a_2 \circ e_2 = a_2$ and $a_1 * e_1 = a_1$

$\Rightarrow a_2 \circ e_2 = f(a_1 * e_1)$

$\Rightarrow a_2 \circ e_2 = f(a_1) \circ f(e_1)$ by isomorphism.
$\qquad\qquad = a_2 \circ f(e_1)$

$\Rightarrow e_2 = f(e_1)$ by cancellation law. *(R6)*

(2) From (1): $e_2 = f(e_1)$

$$\Rightarrow a_2 \circ a_2^{-1} = f(a_1 * a_1^{-1})$$
$$= f(a_1) \circ f(a_1^{-1}) \text{ by isomorphism}$$
$$= a_2 \circ f(a_1^{-1}) \text{ [since } a_2 = f(a_1)\text{]}$$ *(R3)*

$\Rightarrow a_2^{-1} = f(a_1^{-1})$ by cancellation. *(A2)*

6. (i) (a)

	Graph 1	Graph 2	Graph 3	Graph 4	Graph 5
Degrees	4,4,4,4,4	2,2,2,2,2	1,3,2,2,2	2,2,4,2,2	3,2,3,2,2
Eulerian?	Yes, ABCD EACEBDA	Yes ABCDEA	No	Yes ABCDECA	No
Hamiltonian?	Yes ABCDE	Yes ABCDE	No	No	Yes ABCDE

(A8), less 2 per incorrect box.

(b) None of the four statements are true. *(A2)*

(c) A connected graph is Eulerian if and only if the degree of every vertex is even. *(A2)*

Since the graph is connected, every vertex appears in the Eulerian circuit. Whenever a vertex is visited it contributes two to the degree of that vertex, one from the edge going in and one from the edge going out. Hence the degree of every vertex is even. *(R4)* for clear explanation.

(ii)

	A	B	C	D	E	F	G	H
A	*0*	4	5	6				
B		*4*	5	6		18		
C			*5*	6	12	18	14	
D				*6*	12	18	13	
E					*12*	15	13	
G						15	*13*	21
F						*15*		20
H								*20*

Any other algorithmic representation is acceptable. One such method may use the diagram and temporary labels, which are transformed into permanent ones as the algorithm is extended.

(A6) for table, less 2 per error.

The shortest path is A, C, E, F, H and its length is 20. *(A3)*
The length of the shortest path from A to G is 13. *(A3)*

(iii) Let G be a connected planar graph that is not a tree which has v vertices, e edges and divides the plane into f faces or regions. Then Euler's formula connecting v, e and f is $v - e + f = 2$. *(A3)*

If $e = 0$ then the graph consists of an isolated vertex and so $v = f = 1$ and therefore $v + f = e + 2 = 2$. Thus the result is true in this case. *(A2)*

(**Note:** Some may take $e = 1$, implying two vertices and one face or one vertex and two faces. Either way $v + f = e + 2 = 3$. Even though the result holds, but the graph is a tree in this case. Students were not penalised for this.)

Now assume that the result is true for all connected planar graphs with $n - 1$ edges and let G be a connected graph, which is not a tree, with n edges. *(A2)*

Choose any edge sequence in G and remove one edge. The remaining graph is planar, connected, and it has $n - 1$ edges, v vertices and $f - 1$ faces. Thus, by the assumption

$$v - (n - 1) + f - 1 = 2$$

and so $v - n + f = 2$
and the result is true for all connected planar graphs with n edges. Hence the formula holds for all connected planar graphs. *(R5)*

7. (i) The mean number of calls is $\dfrac{\sum fx}{\sum f(x)} = \dfrac{105}{40} = 2.625$ *(M1) (A1)*

The null hypothesis H_0 is that the distribution of calls received is a Poisson distribution with a mean of 2.625. *(A1)*

The expected frequencies are given by $40 \times p(X = x)$ and so the observed and expected values are

O	1	8	12	7	8	4	0
E	2.9	7.61	9.98	8.74	5.73	3.01	2.04

The first and last two entries are combined as they are each less than 5 but the sum is greater than 5. *(A3)*

Then

$$\chi^2 = \sum \frac{(O - E)^2}{E} = 2.09$$ *(A2)*

At the 5% level of significance and with 3 degrees of freedom the critical value of χ^2 is 7.81. *(A2)*

Since the calculated value is less than this we do not reject H_0 and so conclude that the distribution of calls received can reasonably be modelled by a Poisson distribution with mean 2.625. *(R2)*

Note: Some students may not combine any entries. The χ^2 values are then 5.29 with 5 degrees of freedom (critical value 11.07). The conclusion is unchanged.

(ii) (a) Assume that diabetic males X_D are $N(\mu_D, \sigma^2)$ and the non diabetic males X are $N(\mu, \sigma^2)$, *i.e.* the variance is the same.

Then $\bar{X}_D$ is $N(\mu_D, \dfrac{\sigma^2}{n_D})$ and $\bar{X}$ is $N(\mu, \dfrac{\sigma^2}{n})$

for n_D diabetics and n non diabetics. *(A1) (A1)*

The difference $\bar{X}_D - \bar{X}$ is $N\left(\mu_D - \mu,\ \sigma^2\left(\dfrac{1}{n_D} + \dfrac{1}{n}\right)\right)$ and so

$$\frac{(\bar{X}_D - \bar{X}) - (\mu_D - \mu)}{\sigma\sqrt{\dfrac{1}{n_D} + \dfrac{1}{n}}} \text{ is } N(0, 1)$$ *(A2)*

Since σ is unknown the estimate assumes common population variance using pooled variance estimate

$$S^2 = \frac{(n_D - 1)s_D^2 + (n - 1)s^2}{(n_D - 1) + (n - 1)}$$ *(A2)*

Then

$$t = \frac{(\bar{X}_D - \bar{X}) - (\mu_D - \mu)}{S\sqrt{\frac{1}{n_D} + \frac{1}{n}}}$$

is a reading of a t distribution with $(n_D - 1) + (n - 1)$ degrees of freedom. *(A2)*

The 95% symmetric confidence interval comes from

$$\left| \frac{(\bar{X}_D - \bar{X}) - (\mu_D - \mu)}{S\sqrt{\frac{1}{n_D} + \frac{1}{n}}} \right| < t_{0.025}(n_D + n - 2)$$

or

$$(\bar{X}_D - \bar{X}) - t_{0.025}S\sqrt{\frac{1}{n_D} + \frac{1}{n}} < (\mu_D - \mu)$$

$$< (\bar{X}_D - \bar{X}) + t_{0.025}S\sqrt{\frac{1}{n_D} + \frac{1}{n}}$$

From the data

$$S^2 = \frac{24(13.92)^2 + 29(12.59)^2}{53} = 174.474 \Rightarrow S = 13.21$$

and $\bar{X}_D - \bar{X} = 7.99$ *(M1) (A1)*

The 95% confidence interval is thus

$$7.99 \pm 2.01 \times 13.21\sqrt{\frac{1}{25} + \frac{1}{30}} = 7.99 \pm 7.19 = [0.80,\ 15.18].$$ *(A2)*

The null hypothesis H_0: $\mu_D = \mu$ or $\mu_D - \mu = 0$ and the alternative two sided hypothesis H_1 is that $\mu_D \neq \mu$. The critical region is $t > t_{0.025}$ (53 *d.f.*).

For the given values,

$$t = \frac{\bar{X}_D - \bar{X}}{S\sqrt{\frac{1}{n_D} + \frac{1}{n}}} = \frac{7.99}{13.21\sqrt{\frac{1}{25} + \frac{1}{30}}} = 2.23$$ *(A1)*

and since this is greater than the given critical value of 2.01 we reject the null hypothesis and accept the alternative hypothesis. *(R1) (A1)*

An alternative argument is that the confidence interval excludes zero and this implies that the null hypothesis should be rejected. Any such solution would require an explanation.

(b) The observed frequencies are

	High	Not High	
Diabetic	43	157	200
Non diabetic	41	259	300
	84	416	500

If the attributes are independent then the probability of high blood pressure is estimated by $\frac{84}{500}$ and the probability of being diabetic is estimated by $\frac{200}{500}$. *(A1) (A1)*

The estimated frequency of diabetic and high blood pressure is then

$$500 \times \frac{84}{500} \times \frac{200}{500} = 33.6$$ *(A2)*

The problem has one degree of freedom, the other expected frequencies obtained by using complements. *(A2)*

The expected frequencies are

	High	Not High	
Diabetic	33.6	166.4	200
Non diabetic	50.4	249.6	300
	84	416	500

(A2)

and then

$$\chi^2 = \sum \frac{(O - E)^2}{E} = (9.4)^2 \left\{ \frac{1}{33.6} + \frac{1}{166.4} + \frac{1}{50.4} + \frac{1}{249.6} \right\}$$

$$= (9.4)^2 \{0.0596\}$$

$$= 5.27$$ *(M2) (A1)*

This exceeds the critical value of 3.84 and so reject the hypothesis that the classifications are independent. *(R1) (A1)*

Note: Yates' correction factor is appropriate for this problem since d.f = 1.

8. (i) (a) Evaluating $p(x) = x^3 - 21 - x^2 - 2 = x^3 - x^2 - 23$ at $x = 1, 2, 3, \ldots$ it is seen that $p(3) < 0$ and $p(4) > 0$. Hence the curves intersect inside the interval [3, 4] and so $m = 3$. *(A3)*

(b) If we use the scheme $x_{n+1} = g(x_n)$ then for convergence the condition

$$|g'(x)| < 1$$

(A2), but may be implicit.

has to be satisfied near the point x^*.

With $g(x) = \sqrt[3]{x^2 + 23}$

we have $g'(x) = \frac{2x}{3}(x^2 + 23)^{-2/3}$. *(A2)*

In the interval [3, 4] $(x^2 + 23) > 32$ and so $(x^2 + 23)^{-2/3} < \frac{1}{32^{2/3}} = \frac{1}{8\sqrt[3]{2}}$ *(A2)*

and so $0 < \frac{2x}{3}(x^2 + 23)^{-2/3} < \frac{8}{3 \times 8\sqrt[3]{2}} = \frac{1}{3\sqrt[3]{2}} < 1.$

Hence the scheme provides a sequence that converges to x^*. *(R2) (A1)*

With $g(x) = \sqrt[2]{x^3 - 23}$

we have $g'(x) = \frac{3x^2}{2}(x^3 - 23)^{-1/2} > \frac{27}{2\sqrt{41}} > 1$

in the interval [3, 4] and so the scheme does not provide a sequence that converges to x^*. *(R2) (A2)*

With $x_0 = 3$ the first scheme yields $x_1 = 3.174802$, $x_2 = 3.210103$, $x_3 = 3.217378$, $x_4 = 3.218883$, $x_5 = 3.219195$. Thus $x^* = 3.219$ *(A2)*

(ii) (a) Given that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^\theta$$

where $0 < \theta < x$ then, since all terms are positive, the sum is greater than any one term and so

$$e^x > \frac{x^n}{n!}$$

(R3)(AG)

(b) With $n = 1$ the above result becomes

$$e^x > x$$

and if we set $x = p\ln t$, $t > 1$ since > 0, we obtain

$$e^{p\ln t} > p\ln t$$ *(R2) (A2)*

But the left hand side is simply t^p and so

$$\ln t < \frac{t^p}{p}.$$ *(R2)(AG)*

(c) Set $p = \frac{1}{4}$ in the above to give

$$\ln t < 4t^{1/4}$$ *(A2)*

and thus if t has the particular values $k = 1, 2, 3, 4, \ldots$, we have

$$\ln k < 4k^{1/4},\ k \in \mathbb{N}.$$ *(A2)*

Now for each term of the series $\sum_{n=1}^{\infty} \frac{\ln n}{1 + n^{3/2}}$

$$\frac{\ln n}{1 + n^{3/2}} < \frac{4n^{1/4}}{1 + n^{3/2}} < \frac{4n^{1/4}}{n^{3/2}} = \frac{4}{n^{5/4}}$$ *(R2) (A2)*

Now the series $\sum_{n=1}^{\infty} \frac{4}{n^{5/4}}$ converges since the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a convergent series for $p > 1$. *(A3)*

Hence, by the comparison test, the series $\sum_{n=1}^{\infty} \frac{\ln n}{1 + n^{3/2}}$ converges. *(R1) (A1)*